Understanding the Quintile Portfolio

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***Abstract*—The heuristic $1/N$ (i.e., equally weighted) portfolio and heuristic quintile portfolio are both popular simple strategies in financial investment. In the $1/N$ portfolio, a fraction of $1/N$ of the wealth is allocated to each of the N available assets. In the quintile portfolio, first the assets are sorted according to some characteristics, e.g., expected returns, and then the strategy equally longs the top 20% (i.e., top quintile) and perhaps shorts the bottom 20% (i.e., bottom quintile). Although they have been criticized for their lack of mathematical justification when proposed by practitioners, they have shown great advantage over more sophisticated portfolios in terms of stable performance and easy deployment. In this paper, we reinterpret the $1/N$ and quintile portfolios as solutions to a mathematically sound robust portfolio optimization under different levels of robustness level in the stocks' characteristics. A variance-adjusted robustness uncertainty set is also proposed, leading to the inverse-volatility portfolios, whose nonzero weights are inversely proportional to their standard deviation.**

***Index Terms*—$1/N$ portfolio, quintile portfolio, robust portfolio design, robust ℓ_1-norm optimization, inverse-volatility portfolio.**

I. Introduction

Modern portfolio theory has been a rapidly developing field since Harry Markowitz's seminal paper [2]. Prior to that, investments were made in a discretionary manner, i.e., based on people's expertise or experience. Markowitz proposed the idea that risk-adverse investors should optimize their portfolio to achieve a tradeoff of maximum expected return and minimum risk. This idea has remained the pillar of modern quantitive finance and portfolio optimization, effectively shaping the empirical investment into a mathematical and scientific art.

Denote by $\mathbf{w} \in \mathbb{R}^N$ the portfolio weights vector on N assets. Markowitz's mean-variance portfolio can be obtained by solving the following problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} - \lambda \boldsymbol{\mu}^T \mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W}, \end{array} \tag{1}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ are the covariance matrix and mean vector of the returns of the N assets, $\mathcal{W}$ denotes the portfolio constraints, and $\lambda \geq 0$ is a parameter striking a balance between the expected return ($\boldsymbol{\mu}^T \mathbf{w}$) and the portfolio risk (defined by the variance $\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$). After such mathematical formulation, more sophisticated portfolio formulations have been developed, e.g., the minimum Conditional Value-at-Risk portfolio [3], the high-order portfolio [4], and the risk parity portfolio [5].

However, the vanilla Markowitz's portfolio and some other more sophisticated portfolios fail to achieve the promised performance when deployed into real financial markets. This is due to many reasons as recognized in the literature, three of which are described next. First, the variance is not a good measure of risk in practice since it penalizes both the unwanted high losses and the desired low losses. Second, the objective of Markowitz's portfolio formulation, i.e., the problem (1), uses a single measure of overall risk. The distribution of risk is ignored here, so that investors are exposed to the unacceptable losses. Last but not least, a reason for the bad performance is the poor knowledge of the true parameters like $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$. Indeed, in practice they have to be measured by some estimation method that suffers from unavoidable estimation errors [6]. A small perturbation on the parameters, especially the mean vector $\boldsymbol{\mu}$, might lead to a significantly different solution [7]. Another issue arises when the sophisticated portfolio optimization problems are nonconvex, making it difficult to obtain their global optimal solutions [8]. Besides, such portfolio optimization problems, although starting from a reasonable and rigorously theoretical motivation, may finally lead to unpredictable optimal solutions. The frequent failure and lack of intuition make the investors hesitate to take these sophisticated portfolios [9].

Robust parameter estimation and robust portfolio optimization are two traditional approaches dealing with estimation error in the parameters [10]–[12]. The classical robust parameter estimation methods include shrinkage estimation [13], Bayesian approach [14], and Black-Litterman model [11]. Besides, assuming data follows a heavy-tailed distribution usually makes the parameter estimates robust to outliers [6], [15]. Taking the structure of covariance matrix into consideration also helps to obtain a more accurate estimate [16]. Robust portfolio optimization directly considers the parameter uncertainty in the problem formulation, i.e., it optimizes the worst-case objective with parameter located in a known uncertainty set. Various kinds of uncertainty sets with different criteria have been studied in the literature [8], [17]. Recently, such robust portfolio optimization approach has been introduced into high-order portfolio [18] and weather derivative portfolio with temperature uncertainty [19]. Although the robust portfolio optimization can alleviate the influence of parameter estimation errors, the complexity of solving the problem increases. Even a global optimal solution is obtained, it may be difficult to gain an intuitive understanding.

Therefore, instead of using these sophisticated portfolios, practitioners prefer some heuristic but common-sense portfolios,

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e.g., the $1/N$ (a.k.a. equally weighted or uniform) portfolio [20] and quintile portfolio [21], [22]. Motivated by the 'naive' diversification for reducing the risk, the $1/N$ portfolio equally allocates a fraction of $1/N$ of wealth to each asset. A comprehensive study has been done to show that sample-based mean-variance portfolios fail to defeat the $1/N$ portfolio for several empirical datasets, and numerous theory-based optimal portfolios cannot consistently outperform it either [20]. The long-only (long-short) quintile portfolio first sorts the assets according to some characteristics and then equally longs the top 20% (and perhaps shorts the bottom 20%) [22]. It is also called the factor investing as it mimics returns of some common factors [21], [23]. The factor model, e.g., Capital Asset Pricing Model (CAPM) [24] and Fama French model [25], was initially introduced to explain stock returns. Some of the factors used in these models are calculated as the excess returns of assets with attractive characteristics. For example, five well-known factors, namely, Value, Low Size, Low Volatility, Momentum, and Quality, may be obtained by ranking the assets according to book-to-price ratio, low market capitalization, low standard deviation, return, and return on equity, respectively [21]. Empirical studies show that these factors have exhibited excess returns above the market [21], [26]. The quintile portfolio based on the momentum measured by the estimated return in the past few months is one of the most famous ones. A study found that about 77% percent of 155 mutual funds were actually using such kind of portfolio over the 1975-1984 period [27]. Interestingly, some investors might prefer the quintile portfolio based on the opposite of short-term return because they believe in short-term reversals [28]. More evidences on the benefits of quintile portfolios are presented in [29]–[32]. It has been a widely debated mystery how these simple portfolios defeat the theory-based portfolios.

In this paper, we propose a mathematically sound robust portfolio optimization problem against the estimation error of the asset characteristics. The rigorous proofs show that its solution reduces to the heuristic $1/N$ and quintile portfolios under different levels of robustness level in the characteristics. We also propose a variance-adjusted uncertainty set for the characteristics. Such uncertainty set will be shown to be related to the inverse-volatility portfolios, whose nonzero weights are inversely proportional to their standard deviation [33], [34].

The remaining sections of this paper are organized as follows. In Section II, we introduce the heuristic $1/N$ portfolio and quintile portfolio with a simple illustration of their performance. In Section III, we show how to cast the heuristic $1/N$ portfolio and quintile portfolio through a robust portfolio design. The quintile inverse-volatility portfolio is derived from a similar manner in Section IV. In Section V, we present some variations of the classical quintile portfolio. Numerical experiments are evaluated in Section VI and, finally, conclusion is given in Section VII.

II. Heuristic Portfolios and Their Surprising Performance

It is widely accepted that diversification is an effective way to reduce risk in asset allocation. Following this intuition of diversification, practitioners have proposed several heuristic portfolios, e.g., the $1/N$ portfolio and quintile portfolio. These portfolios are much simpler than the Markowitz's portfolio but actually work well in the real markets. The $1/N$ portfolio allocates a fraction of $1/N$ of wealth to each asset:

$$\mathbf{w} = \frac{1}{N}\mathbf{1}_N. \tag{2}$$

It does not require any information from the market. Different from that, the quintile portfolio sorts the assets from top to bottom based on some information from market, e.g., the estimated expected return. Then the quintile portfolio equally longs the top 20% (i.e., top quintile):

$$\mathbf{w} = \frac{1}{N/5} \times [\underbrace{1,\ldots,1}_{20\%},\underbrace{0,\ldots,0}_{80\%}]^T. \tag{3}$$

The bottom 20% (i.e., bottom quintile) can also be equally shorted if the short selling operation is allowed, i.e.,

$$\mathbf{w} = \frac{1}{2\,N/5} \times [\underbrace{1,\ldots,1}_{20\%},\underbrace{0,\ldots,0}_{60\%},\underbrace{-1,\ldots,-1}_{20\%}]^T. \tag{4}$$

The two quintile portfolios are specifically called the long-only quintile portfolio and long-short quintile portfolio, respectively.

A comprehensive study has been done to show that sample-based mean-variance portfolios fail to defeat the $1/N$ portfolio for several empirical datasets, and numerous theory-based optimal portfolios cannot consistently outperform it either [20]. More evidences have been provided to show the profitability of the quintile portfolio [29]–[32]. We perform a backtest to illustrate the surprising performance of these heuristic portfolios. We include some classical theory-based portfolios, namely, the global minimum variance portfolio (GMVP), the mean-variance portfolio (MVP), the global maximum return portfolio (GMRP), and the maximum Sharpe ratio portfolio (MSRP). The problem formulations of these portfolios are given in Table I. We consistently use the no shorting constraint, i.e., $\mathcal{W} = \{\mathbf{w} | \mathbf{w} \geq \mathbf{0}, \mathbf{1}^T\mathbf{w} = 1\}$, for all the portfolios to ensure fairness of comparison. Actually, the GMVP, MVP, and GMRP are special cases of general Markowitz's mean-variance portfolio, i.e., the problem (1), with λ being 0, $\frac{1}{2}$, and $+\infty$. We use the historical daily price data of stocks from Standard &

TABLE I
The Detailed Problem Formulation of Several Theory-Based Portfolios

Name	Objective	Constraint
GMVP	$\underset{\mathbf{w}}{\text{minimize}} \quad \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$	$\mathbf{w} \geq \mathbf{0}$, $\mathbf{1}^T\mathbf{w} = 1.$
MVP	$\underset{\mathbf{w}}{\text{minimize}} \quad \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} - \frac{1}{2}\boldsymbol{\mu}^T\mathbf{w}$	
GMRP	$\underset{\mathbf{w}}{\text{maximize}} \quad \boldsymbol{\mu}^T\mathbf{w}$	
MSRP	$\underset{\mathbf{w}}{\text{maxmize}} \quad \dfrac{\boldsymbol{\mu}^T\mathbf{w}}{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$	

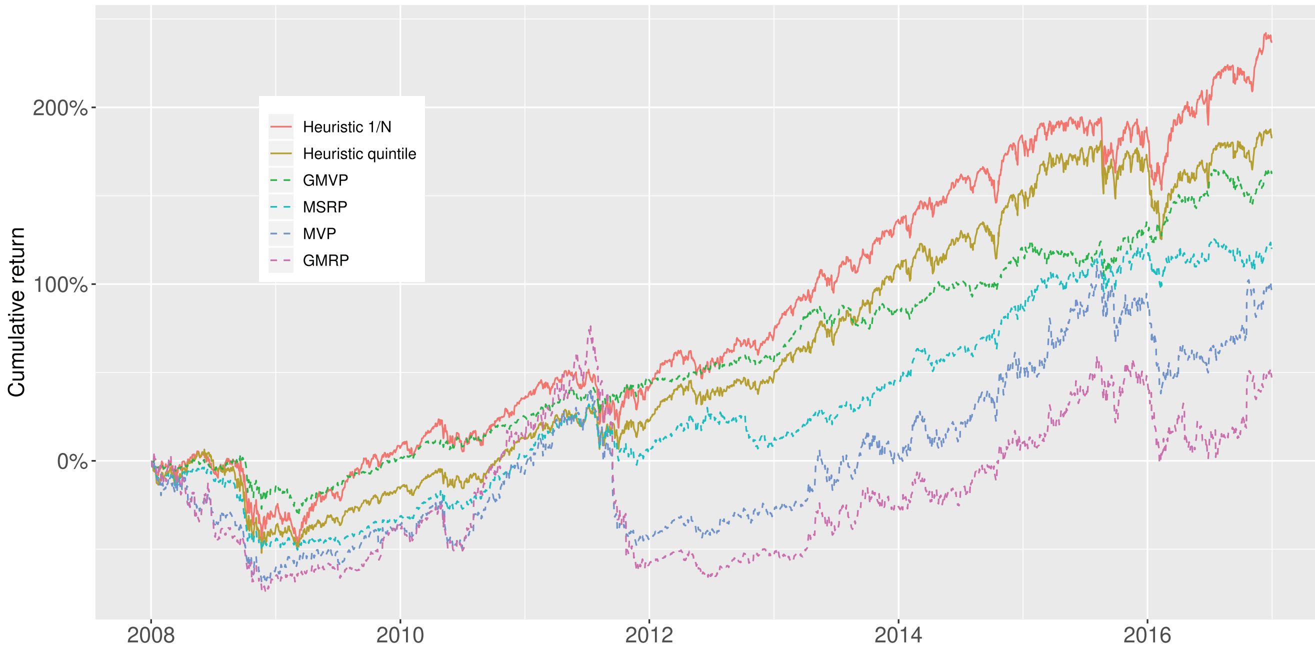

Fig. 1. Cumulative returns of theory-based portfolios and heuristic portfolios using S&P 500 stocks from 2008 to 2016.

Poor (S&P) 500 Index components from 2008 to 2016. The parameter $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated by the sample mean and sample covariance matrix using the data of past 4 years. We reoptimize these portfolios every 3 months and ignore the transaction cost. The cumulative return of portfolios are shown in Fig. 1. It is significantly clear that the heuristic $1/N$ portfolio and the quintile portfolio outperform all theory-based portfolios.

III. Interpreting the $1/N$ and Quintile Portfolios as Robust Portfolio Designs

Considering the fact that we only have the estimation of the asset characteristics, a robust optimization is useful to make the problem robust to parameter errors of assets' characteristics. We are particularly interested in the worst-case robust optimization, which optimizes the worst-case objective when the the true parameter is assumed to lie in an known uncertainty set centered around the estimated value [11]. Without loss of generalization, we interpret the quintile portfolio based on the estimated expected return. It is of no difficulty to replace the expected return with any other characteristic, while the conclusion is still valid.

A. Mean Vector Uncertainty Set

There exists many common uncertainty sets for the mean vector. For example, the box constraint considers the true $\boldsymbol{\mu}$ is element-wise lower and upper bounded, i.e., $\boldsymbol{\mu}_l \leq \boldsymbol{\mu} \leq \boldsymbol{\mu}_u$ [8], [11]. The ellipsoidal constraint assumes the true $\boldsymbol{\mu}$ is located in an ellipsoid around a reference $\bar{\boldsymbol{\mu}}$ as $\{\boldsymbol{\mu} | (\boldsymbol{\mu} - \bar{\boldsymbol{\mu}})^T \mathbf{S}^{-1} (\boldsymbol{\mu} - \bar{\boldsymbol{\mu}}) \leq 1\}$, where $\mathbf{S}$ is usually collinear with $\boldsymbol{\Sigma}$ [11]. The portfolio optimization problem under the above two uncertainty sets has been well developed. But neither of them can help our purpose. In this paper, we consider that $\boldsymbol{\mu}$ lies on an ℓ_1-norm ball around the estimated $\hat{\boldsymbol{\mu}}$, i.e., the uncertainty set is

$$\mathcal{S} = \{\hat{\boldsymbol{\mu}} + \mathbf{e}, \|\mathbf{e}\|_1 \leq \epsilon\}. \tag{5}$$

To simplify the following discussion, we assume without loss of generality that the elements of $\hat{\boldsymbol{\mu}}$ have been sorted in nonincreasing order so that $\hat{\mu}_1 \geq \hat{\mu}_2 \geq \cdots \geq \hat{\mu}_N$.

B. What Formulations Are the $1/N$ Portfolio and Long-Only Quintile Portfolio Solving?

In this section, we show how to derive the naive $1/N$ portfolio as a robust portfolio. First of all, we introduce a technical assumption that says ϵ cannot be exactly equal to some discrete points, the number of which is no more than N. In practice, due to the estimation errors in $\hat{\boldsymbol{\mu}}$, this condition is satisfied with probability 1.

Assumption 1: $\epsilon \notin \mathcal{F}_1 = \{\sum_{i=1}^{k} (\hat{\mu}_i - \hat{\mu}_k) \big| k = 1, \ldots, N\}$.

The robust best characteristic portfolio problem with no shorting constraint is formulated as:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \min\limits_{\boldsymbol{\mu} \in \mathcal{S}} \boldsymbol{\mu}^T \mathbf{w} \\ \text{subject to} & \mathbf{w} \geq \mathbf{0}, \ \mathbf{1}^T \mathbf{w} = 1. \end{array} \tag{6}$$

Lemma 1: The optimal objective of the problem $\min_{\boldsymbol{\mu} \in \mathcal{S}} \boldsymbol{\mu}^T \mathbf{w}$ is $p^\star = \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \|\mathbf{w}\|_\infty$.

Proof: Considering $\mathcal{S}$ in (5), the above problem becomes

$$\underset{\|\mathbf{e}\|_1 \leq \epsilon}{\text{minimize}} \quad \hat{\boldsymbol{\mu}}^T \mathbf{w} + \mathbf{e}^T \mathbf{w}. \tag{7}$$

It is straightforward that the worst-case error $\mathbf{e}$ has to put all the weight on the largest element of $\mathbf{w}$. Therefore, the optimal objective is $\hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \max_i |w_i|$, i.e., $p^\star = \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \|\mathbf{w}\|_\infty$. ■

With usage of Lemma 1, problem (6) is equivalent to

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon\|\mathbf{w}\|_\infty \\ \text{subject to} & \mathbf{w} \geq \mathbf{0},\ \mathbf{1}^T\mathbf{w} = 1. \end{array} \tag{8}$$

Furthermore, it can be reformulated in epigraph form [35] as

$$\begin{array}{ll} \underset{\mathbf{w},t}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon t \\ \text{subject to} & \mathbf{0} \leq \mathbf{w} \leq t\mathbf{1},\ \mathbf{1}^T\mathbf{w} = 1 \end{array} \tag{9}$$

Lemma 2: The optimal solution to problem (9), under Assumption 1, is

$$\mathbf{w}^\star = \left[\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0\right]^T, \tag{10}$$

whose first m elements are $1/m$ and the rest are 0, where m is the maximum $k \in \{1, \ldots, N\}$ satisfying $\sum_{i=1}^k (\hat{\mu}_i - \hat{\mu}_k) < \epsilon$.

Proof: The Lagrangian of problem (9) is given by:

$$\begin{aligned} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \eta) &= \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon t + \boldsymbol{\alpha}^T\mathbf{w} + \boldsymbol{\beta}^T(t\mathbf{1} - \mathbf{w}) \\ &\quad + \eta\left(1 - \mathbf{1}^T\mathbf{w}\right) \end{aligned}$$

Then, we write the KKT conditions as:

$$\begin{aligned} &\mathbf{1}^T\mathbf{w} = 1,\ \mathbf{0} \leq \mathbf{w} \leq t\mathbf{1}, \\ &\boldsymbol{\alpha} \geq 0,\ \boldsymbol{\beta} \geq 0, \\ &\boldsymbol{\alpha}^T\mathbf{w} = \mathbf{0},\ \boldsymbol{\beta}^T(t\mathbf{1} - \mathbf{w}) = \mathbf{0}, \\ &\frac{\partial L}{\partial \mathbf{w}} = \hat{\boldsymbol{\mu}} + \boldsymbol{\alpha} - \boldsymbol{\beta} - \eta\mathbf{1} = 0,\ \frac{\partial L}{\partial t} = \mathbf{1}^T\boldsymbol{\beta} - \epsilon = 0. \end{aligned}$$

According to complementary slackness, if $w_i > 0$, then $\alpha_i = 0$ and $\beta_i = \hat{\mu}_i - \eta \geq 0$, and further $w_i = t$ when $\hat{\mu}_i > \eta$. In contrast, if $w_i = 0$, then we have $\beta_i = 0$ (since it must be that $t > 0$) and $\alpha_i = \eta - \hat{\mu}_i \geq 0$. Equivalently, we can write the following expressions for α_i, β_i, and w_i w.r.t. η as:

$$\begin{aligned} \alpha_i &= (\eta - \hat{\mu}_i)^+, \\ \beta_i &= (\hat{\mu}_i - \eta)^+, \\ w_i &= \begin{cases} t & \hat{\mu}_i > \eta \\ 0 \leq w_i \leq t & \hat{\mu}_i = \eta, \\ 0 & \hat{\mu}_i < \eta \end{cases} \end{aligned} \tag{11}$$

where $(x)^+ = \max(0, x)$. Then, we can easily apply $\mathbf{1}^T\boldsymbol{\beta} = \sum_{i=1}^N (\hat{\mu}_i - \eta)^+ = \epsilon$ to find the $\eta^\star$, which is also known as the classical water-filling solution [36]. According to the condition $\epsilon \notin \mathcal{F}_1$, we have $\hat{\mu}_i \neq \eta^\star$, which means $w_i^\star = \{0, t\}$ is always satisfied. Recall that $\hat{\mu}_1 \geq \hat{\mu}_2 \geq \cdots \geq \hat{\mu}_N$, then the $\mathbf{w}^\star = [t, \ldots, t, 0, \ldots, 0]^T$ is the optimal solution to problem (9). Denote by m the number of non-zero elements in $\mathbf{w}^\star$, it is easy to find that m is equal to the maximum $k \in \{1, \ldots, N\}$ satisfying $\sum_{i=1}^k (\hat{\mu}_i - \hat{\mu}_k) < \epsilon$. Finally considering $\mathbf{1}^T\mathbf{w} = 1$, we then have $t = 1/m$.

According to Lemma 2, the optimal portfolio to problem (6) equally longs the assets with most attractive (estimated) characteristics. The number of active assets, i.e., those with non-zero weights, is determined by the estimation error level ϵ. The larger ϵ, the more active assets. The following two corollaries explain the condition of ϵ, under which the resulting portfolio will be exactly the $1/N$ portfolio or the quintile portfolio. Fig. 2 illustrates the relations between different levels of estimation uncertainty and the returning portfolio. ■

Corollary 3: If $\epsilon > \sum_{i=1}^N (\hat{\mu}_i - \hat{\mu}_N)$, then the optimal solution of problem (6) is exactly the $1/N$ portfolio.

Corollary 4: If $\sum_{i=1}^q (\hat{\mu}_i - \hat{\mu}_q) < \epsilon < \sum_{i=1}^{q+1} (\hat{\mu}_i - \hat{\mu}_{q+1})$, where q is the number corresponding to 20% of the assets, then the optimal solution of problem (6) is exactly the long-only quintile portfolio.

C. What Formulation is the Long-Short Quintile Portfolio Solving?

First of all, we introduce a technical assumption that says ϵ cannot be exactly equal to some discrete points, the number of which is no more than $\lfloor \frac{N}{2} \rfloor$. In practice, due to the estimation errors in $\hat{\boldsymbol{\mu}}$, this condition is satisfied with probability 1.

Assumption 2: $\epsilon \notin \mathcal{F}_2 = \{\sum_{i=1}^k (\hat{\mu}_i - \hat{\mu}_k) + \sum_{i=N-k+1}^N (\hat{\mu}_{N-k+1} - \hat{\mu}_i) \Big| k = 1, \ldots, \lfloor \frac{N}{2} \rfloor\}$.

The robust best characteristic portfolio problem with dollar-neutral constraint is formulated as:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \underset{\boldsymbol{\mu} \in \mathcal{S}}{\min}\ \boldsymbol{\mu}^T\mathbf{w} \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 0,\ \|\mathbf{w}\|_1 = 1. \end{array} \tag{12}$$

According to Lemma 1, it is equivalent to:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon\|\mathbf{w}\|_\infty \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 0,\ \|\mathbf{w}\|_1 = 1, \end{array} \tag{13}$$

which can be rewritten in epigraph form as

$$\begin{array}{ll} \underset{\mathbf{w},t}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon t \\ \text{subject to} & \mathbf{1}^T\mathbf{w} = 0,\ \|\mathbf{w}\|_1 = 1, \\ & -t\mathbf{1} \leq \mathbf{w} \leq t\mathbf{1}. \end{array} \tag{14}$$

Obviously, problem (14) is non-convex because of the ℓ_1-norm equality constraint. To tackle such problem, we need first explore some properties of its optimal solution.

Lemma 5: There exist an optimal solution $\mathbf{w}^\star$ to problem (14) that satisfies the condition:

$$w_i^\star = -w_{N+1-i}^\star \geq 0,\ i = 1, \ldots, \left\lfloor \frac{N+1}{2} \right\rfloor. \tag{15}$$

Proof: Please refer to Appendix A. ■

Lemma 5 tells us that the optimal weight of the asset with i-th highest $\hat{\mu}$ should be exactly opposite to optimal weight of the asset with i-th lowest $\hat{\mu}$. Besides, for such a pair, the optimal weight of the asset with higher $\hat{\mu}$ is nonnegative. Note that Lemma 5 becomes a necessary condition for optimality when $\hat{\mu}_1 > \hat{\mu}_2 > \cdots > \hat{\mu}_N$, i.e., the inequalities are strict.

Using Lemma 5, we can simplify problem (14) by optimizing only the left half elements of $\mathbf{w}$, denoted by $\check{\mathbf{w}}$, as

$$\begin{array}{ll} \underset{\check{\mathbf{w}}}{\text{maximize}} & \mathbf{d}^T\check{\mathbf{w}} - \epsilon\|\check{\mathbf{w}}\|_\infty \\ \text{subject to} & \check{\mathbf{w}} \geq \mathbf{0},\ \mathbf{1}^T\check{\mathbf{w}} = \frac{1}{2}, \end{array} \tag{16}$$

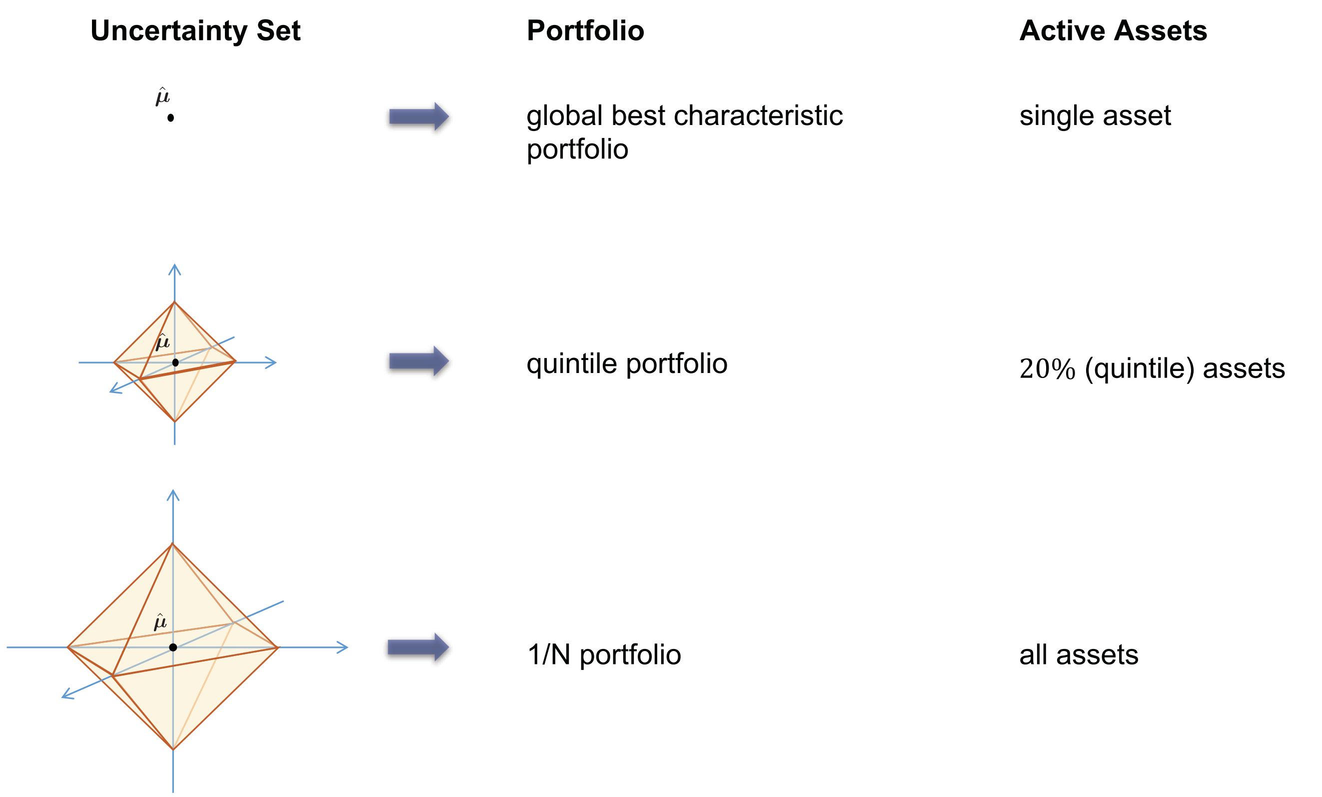

Fig. 2. Different levels of estimation uncertainty and the resulting portfolio.

where $\mathbf{d}$ is a vector of length $\lceil (N-1)/2 \rceil$ with $d_i = \hat{\mu}_i - \hat{\mu}_{N+1-i}$. It can be easily solved by Lemma 2. Then the optimal solution to problem (12) can be easily obtained as the following lemma.

Lemma 6: An optimal solution to problem (12), under Assumption 2, is

$$\mathbf{w}^{\star} = \left[\frac{1}{2\,m}, \ldots, \frac{1}{2\,m}, 0, \ldots, 0, -\frac{1}{2\,m}, \ldots, -\frac{1}{2\,m}\right]^T, \quad (17)$$

whose top and bottom m elements are non-zero, where m is the maximum $k \in \{1, \ldots, \lfloor \frac{N}{2} \rfloor\}$ satisfying $\sum_{i=1}^{k}(\hat{\mu}_i - \hat{\mu}_k) + \sum_{j=N-k+1}^{N}(\hat{\mu}_{N-k+1} - \hat{\mu}_j) < \epsilon$.

Proof: The proof directly follows the combination of Lemma 2 and Lemma 5. ■

According to Lemma 6, the optimal portfolio to problem (12) longs the assets with most attractive (estimated) characteristics and shorts the exactly same number of the assets with least attractive (estimated) characteristics. The weight allocated to each active asset is exactly the same. The number of active assets is also determined by the estimation error level ϵ. The larger ϵ, the more active assets.

Corollary 7: If $f(q) < \epsilon < f(q+1)$, where $f(m) = \sum_{i=1}^{m}(\hat{\mu}_i - \hat{\mu}_m) + \sum_{j=N-m+1}^{N}(\hat{\mu}_{N-m+1} - \hat{\mu}_j)$ and q is the number corresponding to 20% of the assets, then the top 20% assets will be long and bottom 20% assets will be short equally, which is exactly the long-short quintile portfolio.

IV. From Equal Weights to Inverse-Volatility Weights

We have shown, in the above section, that the $1/N$ portfolio and long-only quintile portfolio can be derived from the robust portfolio designs. However, the uncertainty set $\mathcal{S}$ might not be convincing enough because it assumes the estimation error can equally affect all elements in $\hat{\boldsymbol{\mu}}$. Compared with that, it is more reasonable to believe that the $\hat{\mu}_i$ with larger variance is likely to suffer larger estimation error. In this section, we propose an improved version of the characteristic vector uncertainty set, which can be used to derive another series of portfolios called inverse-volatility portfolio.

A. Long-Only Quintile-IV Portfolio

Based on the above analysis, we propose to adjust $\mathcal{S}$ using the volatility information as

$$\mathcal{A}_1 = \{\hat{\boldsymbol{\mu}} + \mathbf{e}, \|\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}\|_1 \leq \epsilon\}, \quad (18)$$

where $\oslash$ is element-wise division and $\hat{\boldsymbol{\sigma}} = [\hat{\sigma}_1, \ldots, \hat{\sigma}_N]^T > \mathbf{0}$ is the estimated standard deviation (i.e., volatility) of the returns of N assets. An example on comparing the shapes of uncertainty sets is given in Fig. 3, where the maximum error on $\hat{\mu}_i$, i.e., the maximum absolute value of e_i, is proportional to $\hat{\sigma}_i$.Similar to Assumption 1, we introduce a technical assumption that says ϵ in $\mathcal{A}_1$ cannot be exactly equal to some discrete points, the number of which is no more than N. In practice, due to the estimation errors in $\hat{\boldsymbol{\mu}}$, this condition is satisfied with probability 1.

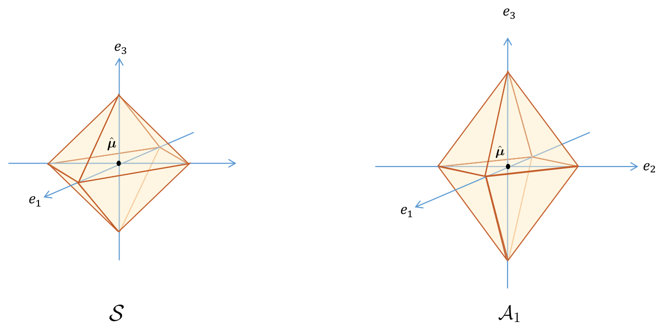

Fig. 3. Comparison of shapes of uncertainty sets $\mathcal{S}$ and $\mathcal{A}_1$ with $\hat{\sigma}_1 < \hat{\sigma}_2 < \hat{\sigma}_3$.

Assumption 3: $\epsilon \notin \{\sum_{i=1}^{k} \frac{\hat{\mu}_i - \hat{\mu}_k}{\hat{\sigma}_i} \big| k = 1, \ldots, N\}$.

The robust best characteristic portfolio problem with no shorting constraint is formulated as:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \underset{\boldsymbol{\mu} \in \mathcal{A}_1}{\min}\ \boldsymbol{\mu}^T \mathbf{w} \\ \text{subject to} & \mathbf{w} \geq \mathbf{0},\ \mathbf{1}^T \mathbf{w} = 1. \end{array} \tag{19}$$

Lemma 8: The optimal objective of the problem $\min_{\boldsymbol{\mu} \in \mathcal{A}_1} \boldsymbol{\mu}^T \mathbf{w}$ is $p^\star = \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \|\hat{\boldsymbol{\sigma}} \odot \mathbf{w}\|_\infty$, where $\odot$ is the Hadamard product.

Proof: The problem $\min_{\boldsymbol{\mu} \in \mathcal{A}} \boldsymbol{\mu}^T \mathbf{w}$ can be rewritten as

$$\underset{\|\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}\|_1 \leq \epsilon}{\text{minimize}} \quad \hat{\boldsymbol{\mu}}^T \mathbf{w} + (\mathbf{e} \oslash \hat{\boldsymbol{\sigma}})^T (\hat{\boldsymbol{\sigma}} \odot \mathbf{w})$$

Using Lemma 1, it is obvious that $p^\star = \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \|\hat{\boldsymbol{\sigma}} \odot \mathbf{w}\|_\infty$. ■

With usage of Lemma 8, problem (19) is equivalent to

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon \|\hat{\boldsymbol{\sigma}} \odot \mathbf{w}\|_\infty \\ \text{subject to} & \mathbf{w} \geq \mathbf{0},\ \mathbf{1}^T \mathbf{w} = 1. \end{array} \tag{20}$$

Furthermore, it can be reformulated in epigraph form as

$$\begin{array}{ll} \underset{\mathbf{w}, t}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T \mathbf{w} - \epsilon t \\ \text{subject to} & \mathbf{w} \geq \mathbf{0},\ \mathbf{1}^T \mathbf{w} = 1,\ \hat{\boldsymbol{\sigma}} \odot \mathbf{w} \leq t\mathbf{1} \end{array} \tag{21}$$

Lemma 9: The optimal solution to problem (19), under Assumption 3, is

$$\mathbf{w}^\star = \left(\sum_{i=1}^{m} \frac{1}{\hat{\sigma}_i} \right)^{-1} \left[\frac{1}{\hat{\sigma}_1}, \ldots, \frac{1}{\hat{\sigma}_m}, 0, \ldots, 0 \right]^T, \tag{22}$$

whose only first m elements are nonzero, where m is the maximum $k \in \{1, \ldots, N\}$ satisfying $\sum_{i=1}^{k} (\hat{\mu}_i - \hat{\mu}_k)/\hat{\sigma}_i < \epsilon$.

Proof: Please refer to Appendix B. ■

It provides the variations of $1/N$ portfolio and long-only quintile portfolio. The optimal portfolio to problem (19) still longs the assets with the most attractive (estimated) characteristics, while the wealth is no longer equally allocated to assets, but in inverse proportion to their standard deviation. We call this portfolio quintile inverse-volatility (quintile-IV) portfolio. Similar to the long-only quintile portfolio, the larger ϵ, the more active assets.

Corollary 10: If $\epsilon > \sum_{i=1}^{N} (\hat{\mu}_i - \hat{\mu}_N)/\hat{\sigma}_i$, then the optimal solution of problem (19) is

$$\mathbf{w}^\star = \left(\sum_{i=1}^{N} \frac{1}{\hat{\sigma}_i} \right)^{-1} \left[\frac{1}{\hat{\sigma}_1}, \ldots, \frac{1}{\hat{\sigma}_N} \right]^T. \tag{23}$$

Corollary 11: If $\sum_{i=1}^{q} (\hat{\mu}_i - \hat{\mu}_q)/\hat{\sigma}_i < \epsilon < \sum_{i=1}^{q+1} (\hat{\mu}_i - \hat{\mu}_{q+1})/\hat{\sigma}_i$, where q is the number corresponding to 20% of the assets, then the optimal solution of problem (19) is

$$\mathbf{w}^\star = \left(\sum_{i=1}^{q} \frac{1}{\hat{\sigma}_i} \right)^{-1} \left[\frac{1}{\hat{\sigma}_1}, \ldots, \frac{1}{\hat{\sigma}_q}, 0, \ldots, 0 \right]^T. \tag{24}$$

B. Long-Short Quintile-IV Portfolio

In this section, we introduce another uncertainty set of $\boldsymbol{\mu}$ as $\boldsymbol{\mu} \in \mathcal{A}_2 = \{\hat{\boldsymbol{\mu}} + \mathbf{e}, \mathbf{1}^T [\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}]_+ \leq \epsilon_+, -\mathbf{1}^T [\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}]_- \leq \epsilon_-\}$, where $[\mathbf{x}]_+ = \max(\mathbf{x}, \mathbf{0})$ and $[\mathbf{x}]_- = \min(\mathbf{x}, \mathbf{0})$. It is slightly different from previous one by splitting it into positive and negative error. Then the robust best characteristic portfolio problem with dollar-neutral constraint can be formulated as:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \underset{\boldsymbol{\mu} \in \mathcal{A}_2}{\min}\ \boldsymbol{\mu}^T \mathbf{w} \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 0,\ \|\mathbf{w}\|_1 = 1. \end{array} \tag{25}$$

It is obvious that for the inner problem of problem (25), $\mathbf{e}^\star(\mathbf{w}) \odot \mathbf{w} \leq \mathbf{0}$ is guaranteed. Then we can separate $\mathbf{w}$ into $\mathbf{w}_+ = [\mathbf{w}]_+$ and $\mathbf{w}_- = [\mathbf{w}]_-$, and uncertainty set $\mathcal{A}_2$ into $\mathcal{A}_2^- = \{\hat{\boldsymbol{\mu}} + \mathbf{e} | \mathbf{e} \leq \mathbf{0}, -\mathbf{1}^T [\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}]_- \leq \epsilon_-\}$ and $\mathcal{A}_2^+ = \{\hat{\boldsymbol{\mu}} + \mathbf{e} | \mathbf{e} \geq \mathbf{0}, \mathbf{1}^T [\mathbf{e} \oslash \hat{\boldsymbol{\sigma}}]_+ \leq \epsilon_+\}$. Then the problem (25) can be rewritten as

$$\begin{array}{ll} \underset{\mathbf{w}_+, \mathbf{w}_-}{\text{minimize}} & \underset{\boldsymbol{\mu} \in \mathcal{A}_2^-}{\min}\ \boldsymbol{\mu}^T \mathbf{w}_+ + \underset{\boldsymbol{\mu} \in \mathcal{A}_2^+}{\min}\ \boldsymbol{\mu}^T \mathbf{w}_- \\ \text{subject to} & \mathbf{w}_+ \geq \mathbf{0},\ \mathbf{1}^T \mathbf{w}_+ = \frac{1}{2}, \\ & \mathbf{w}_- \leq \mathbf{0},\ \mathbf{1}^T \mathbf{w}_- = -\frac{1}{2}, \\ & \mathbf{w}_+^\star \odot \mathbf{w}_-^\star = \mathbf{0}. \end{array} \tag{26}$$

For temporary, we can decouple the variables and obtain the two separated problems by ignoring the constraints $\mathbf{w}_+^\star \odot \mathbf{w}_-^\star = \mathbf{0}$

$$\begin{array}{ll} \underset{\mathbf{w}_+}{\text{maximize}} & \underset{\boldsymbol{\mu} \in \mathcal{A}_2^-}{\min}\ \boldsymbol{\mu}^T \mathbf{w}_+ \\ \text{subject to} & \mathbf{w}_+ \geq \mathbf{0},\ \mathbf{1}^T \mathbf{w}_+ = \frac{1}{2}, \end{array} \tag{27}$$

$$\begin{array}{ll} \underset{\mathbf{w}_-}{\text{maximize}} & \underset{\boldsymbol{\mu} \in \mathcal{A}_2^+}{\min}\ \boldsymbol{\mu}^T \mathbf{w}_- \\ \text{subject to} & \mathbf{w}_- \leq \mathbf{0},\ \mathbf{1}^T \mathbf{w}_- = -\frac{1}{2} \end{array} \tag{28}$$

Remark 12: Denote by $\mathbf{w}_+^\star$ and $\mathbf{w}_-^\star$ the optimal solutions to problem (27) and (28). If $\mathbf{w}_+^\star \odot \mathbf{w}_-^\star = \mathbf{0}$, then $\mathbf{w}^\star = \mathbf{w}_+^\star + \mathbf{w}_-^\star$ is the optimal solution to the problem (25).

Assumption 4: $\epsilon^- \notin \{\sum_{i=1}^{k} \frac{(\hat{\mu}_i - \hat{\mu}_k)}{\hat{\sigma}_i} \big| k = 1, \ldots, N\}$ and $\epsilon^+ \notin \{\sum_{j=1}^{k} \frac{(\hat{\mu}_{N-k+1} - \hat{\mu}_{N-j+1})}{\hat{\sigma}_{N-j+1}} \big| k = 1, \ldots, N\}$.

Using Lemma 9, it is easy to find that

$$\mathbf{w}_{+}^{\star}=\left(\sum_{i=1}^{m_1}\frac{1}{\hat{\sigma}_i}\right)^{-1}\left[\frac{1}{\hat{\sigma}_1},\ldots,\frac{1}{\hat{\sigma}_{m_1}},0,\ldots,0\right]^T$$

$$\mathbf{w}_{-}^{\star}=\left(\sum_{j=N-m_2+1}^{N}\frac{1}{\hat{\sigma}_i}\right)^{-1}\left[0,\ldots,0,\frac{-1}{\sigma_{N-m_2+1}},\ldots,\frac{-1}{\sigma_N}\right]^T$$

where the first m_1 and the last m_2 elements in $\mathbf{w}_{+}^{\star}$ and $\mathbf{w}_{-}^{\star}$ are nonzero, with m_1 be the maximum $k\in\{1,\ldots,N\}$ satisfying $\sum_{i=1}^{k}\frac{(\hat{\mu}_i-\hat{\mu}_k)}{\hat{\sigma}_i}<\epsilon^-$ and m_2 be the maximum $k\in\{1,\ldots,N\}$ satisfying $\sum_{j=1}^{k}\frac{(\hat{\mu}_{N-k+1}-\hat{\mu}_{N-j+1})}{\hat{\sigma}_{N-j+1}}<\epsilon^+$. It is possible that if ϵ_+ and ϵ_- are under proper control, then $\mathbf{w}_{+}^{\star}\odot\mathbf{w}_{-}^{\star}=\mathbf{0}$ can be satisfied.

Corollary 13: If $\sum_{i=1}^{q}(\hat{\mu}_i-\hat{\mu}_q)/\hat{\sigma}_i<\epsilon_-<\sum_{i=1}^{q+1}(\hat{\mu}_i-\hat{\mu}_{q+1})/\hat{\sigma}_i$ and $\sum_{j=1}^{q}\frac{(\hat{\mu}_{N-q+1}-\hat{\mu}_{N-j+1})}{\hat{\sigma}_{N-j+1}}<\epsilon_+<\sum_{j=1}^{q+1}\frac{(\hat{\mu}_{N-q}-\hat{\mu}_{N-j+1})}{\hat{\sigma}_{N-j+1}}$, where q is the number corresponding to 20% of the assets, then the optimal solution of problem (25) is

$$\mathbf{w}^{\star}=\left[\frac{t_1}{\hat{\sigma}_1},\ldots,\frac{t_1}{\hat{\sigma}_q},0,\ldots,0,\frac{-t_2}{\sigma_{N-q+1}},\ldots,\frac{-t_2}{\sigma_N}\right]^T$$

where $t_1=1/\sum_{i=1}^{q}\hat{\sigma}_i^{-1}$ and $t_2=1/\sum_{i=N-q+1}^{N}\hat{\sigma}_i^{-1}$.

V. Variations of Classical Quintile Portfolio

A. Market Exposure Constraint

Another popular constraint in the portfolio optimization problem is market exposure constraint. Instead of using constraint $\mathbf{1}^T\mathbf{w}=1$, some investors would choose $\boldsymbol{\beta}^T\mathbf{w}=\beta_0$, where $\boldsymbol{\beta}\in\mathbb{R}^N$ is the assets exposure to the market and β_0 is the target portfolio exposure to the market [37]. Here we assume that $\boldsymbol{\beta}>\mathbf{0}$ and $\beta_0>0$, which is quite common in real markets. Then the corresponding robust best characteristic portfolio problem with no shorting constraint is formulated as

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \min_{\boldsymbol{\mu}\in\mathcal{S}}\boldsymbol{\mu}^T\mathbf{w}\\ \text{subject to} & \mathbf{w}\geq\mathbf{0},\ \boldsymbol{\beta}^T\mathbf{w}=\beta_0.\end{array}\tag{29}$$

According to Lemma 1, it is equivalent to

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w}-\epsilon\|\mathbf{w}\|_\infty\\ \text{subject to} & \mathbf{w}\geq\mathbf{0},\ \boldsymbol{\beta}^T\mathbf{w}=\beta_0.\end{array}\tag{30}$$

Denoting $\tilde{\mathbf{w}}=(\boldsymbol{\beta}/\beta_0)\odot\mathbf{w}$, problem (30) can be rewritten as

$$\begin{array}{ll}\underset{\tilde{\mathbf{w}}}{\text{maximize}} & \tilde{\boldsymbol{\mu}}^T\tilde{\mathbf{w}}-\epsilon\|\left(\boldsymbol{\beta}/\beta_0\right)^{-1}\odot\tilde{\mathbf{w}}\|_\infty\\ \text{subject to} & \tilde{\mathbf{w}}\geq\mathbf{0},\ \mathbf{1}^T\tilde{\mathbf{w}}=1,\end{array}\tag{31}$$

where $\tilde{\boldsymbol{\mu}}=\hat{\boldsymbol{\mu}}\oslash(\boldsymbol{\beta}/\beta_0)$ is the element-wise scaled estimated characteristic vector. The optimal solution to problem (31) can be easily obtained using Lemma 9. The assets with highest value of $\tilde{\mu}$, i.e., characteristic-beta ratio, will be active. Let $\tilde{w}_i>0$ belong to an active asset, we have $\tilde{w}_i^{\star}\propto\beta_i/\beta_0$. Considering $w_i=\tilde{w}_i/(\beta_i/\beta_0)$, we can easily get that the non-zero elements in $\mathbf{w}^{\star}$ are equally weighted, which admits a similar form as the optimal solution to problem (6). But it ranks the assets according to the characteristic-beta ratio instead of the simple characteristic vector. Assuming assets have been sorted according to $\tilde{\boldsymbol{\mu}}$, the number of active assets is given by the relative size of ϵ and $\sum_{i=1}^{k}(\tilde{\mu}_i-\tilde{\mu}_k)/(\beta_0/\beta_i)=\sum_{i=1}^{k}(\hat{\mu}_i-\frac{\beta_i}{\beta_k}\tilde{\mu}_k)$, which is not related to β_0. Therefore, β_0 only affects the size of the equal weight.

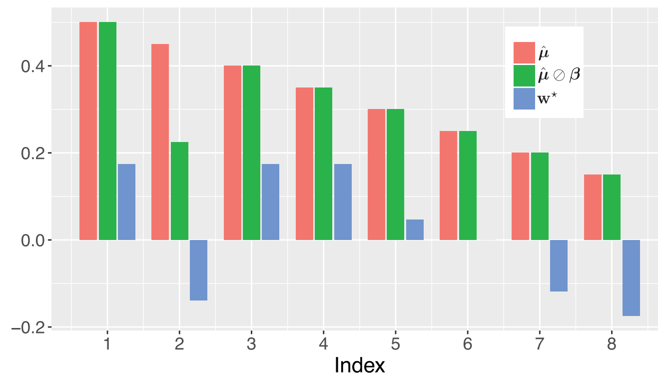

Fig. 4. An example of optimal solution to problem (33).

B. Market-Neutral Constraint

Instead of the dollar-neutral constraint, some investors may prefer the market-neutral constraint as it can eliminate market risk exposure [38]. The corresponding robust best characteristic portfolio problem with market-neutral constraint is

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \min_{\boldsymbol{\mu}\in\mathcal{S}}\boldsymbol{\mu}^T\mathbf{w}\\ \text{subject to} & \|\mathbf{w}\|_1=1,\ \boldsymbol{\beta}^T\mathbf{w}=0.\end{array}\tag{32}$$

According to Lemma 1, it is equivalent to

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w}-\epsilon\|\mathbf{w}\|_\infty\\ \text{subject to} & \|\mathbf{w}\|_1=1,\ \boldsymbol{\beta}^T\mathbf{w}=0.\end{array}\tag{33}$$

We notice that the appealing structure of the long-short quintile portfolio is broken down in this case. A simple numerical example is given in Fig. 4, where we let the elements in $\hat{\boldsymbol{\mu}}$ change from 0.5 to 0.15 with equal spacing of 0.05, $\epsilon=0.4$, and $\boldsymbol{\beta}=[1,2,1,\ldots]^T$. First we can observe that the numbers of assets allocated with positive and negative weights are not the same. Also, the active assets do not follow from a simple characteristic ranking or characteristic-beta ranking as in the long-only case. Besides, the weights are not of the same size any more. Nevertheless, the fact that the solution does not fit the quintile interpretation does not mean that one cannot solve problem (33).

C. Zero Active Sector Exposure Constraint

Consider the zero active sector exposure constraint in the optimization problem as $\mathbf{1}_{n_k}^T\mathbf{w}_k=0, k=1,\ldots,K$, where K is the number of sectors, n_k is the number of assets in the k-th sector, and $\mathbf{w}_k\in\mathbb{R}^{n_k}$ is the weights of assets in k-th sector [39]. The corresponding robust best characteristic portfolio problem with dollar-neutral constraint becomes:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \min_{\boldsymbol{\mu}\in\mathcal{S}} \boldsymbol{\mu}^T\mathbf{w}\\ \text{subject to} & \|\mathbf{w}\|_1 = 1,\\ & \mathbf{1}_{n_k}^T\mathbf{w}_k = 0, k = 1,\ldots,K.\end{array} \tag{34}$$

According to Lemma 1, it is equivalent to:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon\|\mathbf{w}\|_\infty\\ \text{subject to} & \|\mathbf{w}\|_1 = 1,\\ & \mathbf{1}_{n_k}^T\mathbf{w}_k = 0, k = 1,\ldots,K.\end{array} \tag{35}$$

The following lemma shows that, in the k-th sector, for a pair of assets consisting of the one with i-th highest $\hat{\mu}$ and the one with i-th lowest $\hat{\mu}$, the optimal weights are exactly opposite and the optimal weight on the asset with higher $\hat{\mu}$ is nonnegative.

Lemma 14: There exist an optimal solution $\mathbf{w}^\star$ to problem (35) that satisfies the condition:

$$w_{k,i}^\star = -w_{k,n_k+1-i}^\star \geq 0,\ i = 1,\ldots,\left\lfloor\frac{n_k+1}{2}\right\rfloor, \tag{36}$$

where $w_{k,i}$ is the weight of the asset with i-th highest $\hat{\mu}$ in k-th sector.

Proof: The proof follows the Appendix A within the k-th sector. Thus it is omitted here. ■

Using Lemma 14, we are aware that, in the optimal solution to problem (35), the positive weights and negative weights follow the one-to-one negative correspondence inside each sector. We can simply pair these assets and denote by $\check{\mathbf{w}} \geq \mathbf{0}$ the positive weights allocated to the assets with higher estimated return in these sector-aware pairs. Then we can transform the problem (35) into

$$\begin{array}{ll}\underset{\check{\mathbf{w}}}{\text{maximize}} & \mathbf{d}_s^T\check{\mathbf{w}} - \epsilon\|\check{\mathbf{w}}\|_\infty\\ \text{subject to} & \check{\mathbf{w}} \geq \mathbf{0},\ \mathbf{1}^T\check{\mathbf{w}} = \frac{1}{2},\end{array} \tag{37}$$

where $\mathbf{d}_s$ is a vector with each element representing difference of $\hat{\mu}$ from a pair of assets. It can also be easily solved by Lemma 2. Note that in this case, the optimal portfolio is still equally weighted, while the ranking is based on a section-aware fashion. The optimal portfolio follows a quintile structure on a sector-basis.

D. Robust Portfolio With General Risk Control

The quintile portfolio has been shown as an optimal solution to a robust portfolio optimization problem against the characteristic uncertainty. However, the risk management is absent from the optimization, while it is actually a crucial element of portfolio design [40]. To enhance the risk control of quintile portfolio, we can introduce the risk management term into our robust portfolio optimization problem:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \min_{\boldsymbol{\mu}\in\mathcal{S}} \boldsymbol{\mu}^T\mathbf{w} - g(\mathbf{w})\\ \text{subject to} & \mathbf{w} \in \mathcal{W},\end{array} \tag{38}$$

where $g(\mathbf{w})$ measures the risk of the portfolio, e.g., the expected variance or risk concentration [41]. In such a case, however, the appealing structure of the quintile or $1/N$ portfolios most likely vanish. We leave the study of the above problem to the future work.

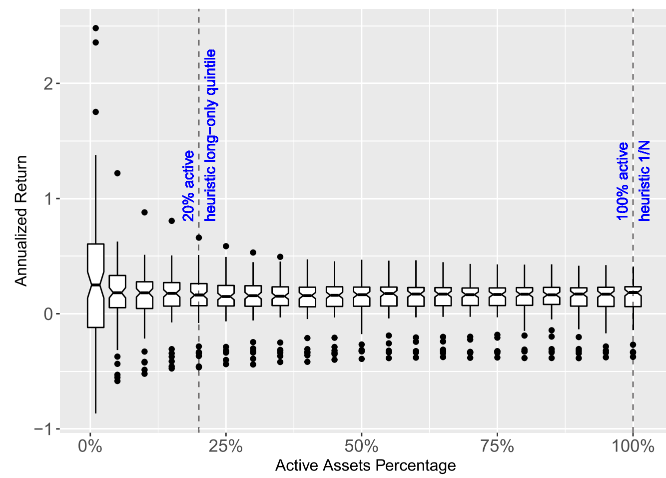

Fig. 5. Annualized return of long-only quintile portfolios based on estimated returns.

VI. Numerical Experiments

In this paper, we have reinterpreted the heuristic $1/N$ and quintile portfolios as solutions to a mathematically sound robust portfolio optimization under different levels of robustness level in the characteristics. The benefits of these heuristic portfolios have been well verified in various markets, e.g., [20], [32]. According to our proposed robust portfolio optimization problem, the heuristic portfolio can be derived from optimizing the worst-case performance. Therefore, we will only focus on providing some evidence on the reasonability of our proposed problem formulation. The experiments are performed with backtests on the quintile portfolio and quintile-IV portfolio.

A. Data and Backtest Description

We first obtain 100 datasets, each of them containing the historical daily price[1] data of 100 stocks over 1000 continuous trading days. The datasets are obtained by randomly choosing from a dataset of 500 stocks, all of them listed in the S&P 500 Index components,[2] from 2004-01-01 to 2018-12-31. Then for each dataset we perform the backtest on a rolling-window basis (see R package portfolioBacktest [42]) with a lookback window length of 252 (one year trading days for training) for obtaining the optimal portfolio and a window length of 22 (one month trading days for testing) for verifying the portfolio performance. To simplify the comparison, we assume a daily rebalancing scheme and ignore the transaction cost.

B. Ranking on Estimated Returns

First, we rank the stocks based on the estimated returns (corresponding to the Momentum factor), which are captured by the sample mean of the return of assets during the lookback window. In Fig. 5, we consider the long-only constraints and show the distribution of annualized returns over 100 dataset for each quintile

[1]Historical daily prices of stocks are available at https://finance.yahoo.com/.
[2]Symbols of S&P 500 Index components are available at https://en.wikipedia.org/wiki/List_of_S%26P_500_companies.

TABLE II
PERFORMANCE OF QUINTILE PORTFOLIOS BASED ON ESTIMATED RETURNS (95% CONFIDENCE INTERVAL)

Constraint	Active Pct.	Type	Annualized Return	Annualized Volatility	Sharpe Ratio	CVaR$_{0.95}$
long-only	20%	quintile	14.73% ± 3.71%	20.29% ± 1.40%	0.93 ± 0.18	2.95% ± 0.22%
		quintile-IV	13.50% ± 3.42%	19.17% ± 1.28%	0.89 ± 0.18	2.79% ± 0.20%
	100%	quintile	14.37% ± 2.97%	18.03% ± 2.00%	1.15 ± 0.20	2.56% ± 0.29%
		quintile-IV	13.45% ± 2.73%	16.60% ± 1.78%	1.17 ± 0.19	2.37% ± 0.26%
long-short	20% + 20%	quintile	−0.52% ± 2.03%	8.30% ± 0.88%	0.13 ± 0.18	1.19% ± 0.14%
		quintile-IV	−0.68% ± 1.90%	7.88% ± 0.84%	0.10 ± 0.19	1.14% ± 0.13%
	50% + 50%	quintile	−0.42% ± 1.09%	4.69% ± 0.49%	0.10 ± 0.18	0.67% ± 0.08%
		quintile-IV	−0.58% ± 0.97%	4.30% ± 0.45%	0.04 ± 0.18	0.62% ± 0.07%

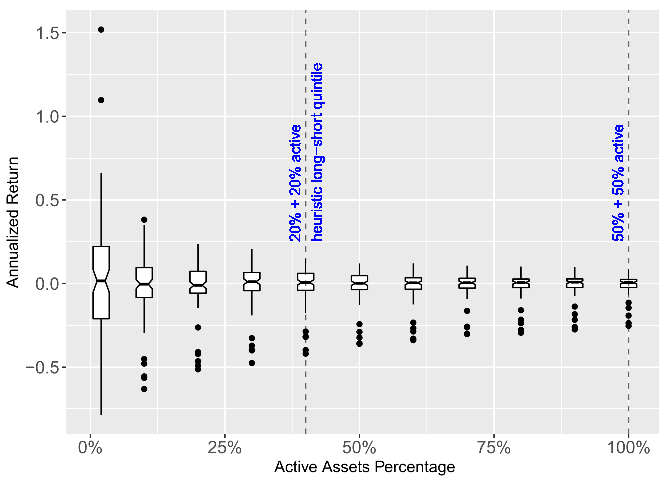

Fig. 6. Annualized return of long-short quintile portfolios based on estimated returns.

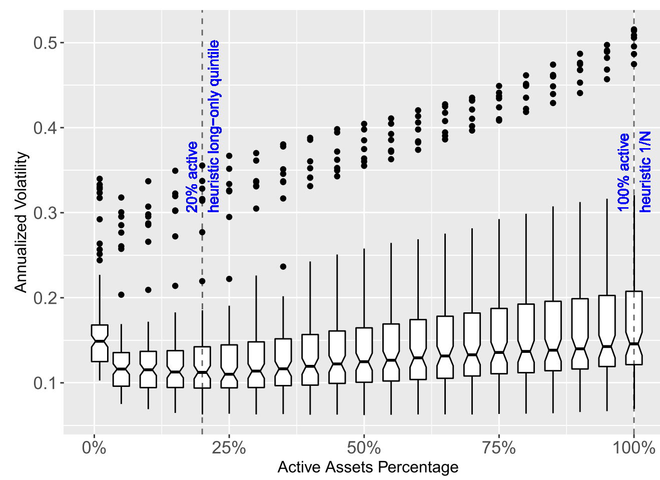

Fig. 7. Annualized volatility of long-only quintile portfolios based on estimated volatilities.

portfolio. Fig. 6 shows the results following the same fashion but on the long-short quintile portfolio. Note that the long-short quintile portfolio shows unsatisfactory profitability due to the financial crisis of 2007-08 [22]. We are particularly interested in the worst-case annualized return as it corresponds with the objective of our proposed problem formulation. If outliers are considered as worst cases, both figures show that the worst-case return can always be improved by activating more assets, which supports the $1/N$ portfolio. If outliers are ignored, however, the best performance of worst-case return appears at around 70% active assets in the long-only case, while still 100% active assets in the long-short case. The results of the quintile-IV portfolio in terms of the annualized return is very similar to results of the quintile portfolio. We report the detailed performance of the quintile and quintile-IV portfolio based on estimated returns in Table II. The backtest results imply that the sample estimation of expected returns might be extremely noisy.

C. Ranking on Estimated Volatilities

Then, we rank the stocks based on the estimated volatilities (stocks with low volatilities are ranking at top positions, corresponding to the Low Volatility factor), which are captured by the sample deviations of the return of assets during the lookback window. We only show the results of long-only quintile portfolio

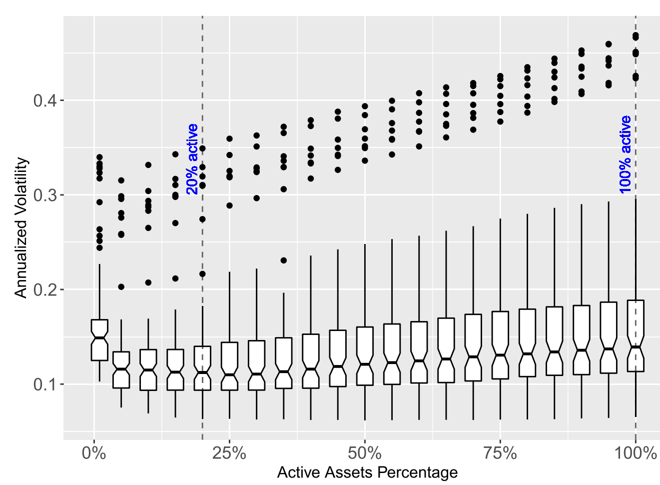

Fig. 8. Annualized volatility of long-only quintile-IV portfolios based on estimated volatilities.

as shorting high volatility stocks does not intuitively reduce the volatility of portfolio return. Fig. 7 and Fig. 8 shows the distribution of annualized volatilities over 100 dataset for each quintile and quintile-IV portfolio. Interestingly, no matter if we ignore the outliers, the the best performance of worst-case volatility appears at around 5% active assets in both figures. Besides, the

TABLE III
PERFORMANCE OF QUINTILE PORTFOLIOS BASED ON ESTIMATED VOLATILITIES (95% CONFIDENCE INTERVAL)

Constraint	Active Pct.	Type	Annualized Return	Annualized Volatility	Sharpe Ratio	CVaR$_{0.95}$
long-only	20%	quintile	11.82% ± 2.36%	13.00% ± 1.20%	1.24 ± 0.20	1.83% ± 0.18%
		quintile-IV	11.81% ± 2.32%	12.85% ± 1.17%	1.24 ± 0.20	1.81% ± 0.18%
	100%	quintile	14.37% ± 2.97%	18.03% ± 2.00%	1.15 ± 0.20	2.56% ± 0.29%
		quintile-IV	13.45% ± 2.73%	16.60% ± 1.78%	1.17 ± 0.19	2.37% ± 0.26%

quintile-IV portfolio seems to outperform the quintile portfolio in terms of the worst-case volatility. The detailed performance of quintile portfolio based on the estimated volatilities is reported in Table III. It implies that the sample estimation of standard deviation might be tolerable.

VII. CONCLUSION

This paper has considered quintile portfolios, both long-only and long-short ones, commonly used by practitioners in financial markets, as well as the $1/N$ portfolio. We have formulated a mathematically meaningful robust best characteristic portfolio design problem, whose solution is a family of quintile portfolios with different active number of assets under different uncertainty level of the stocks' characteristics. Based on that, we have also shown how to derive the inverse-volatility portfolio by proposing a variance-adjusted uncertainty set of the characteristic vector. It bridges the gap between the practitioner heuristic quintile portfolio and Markowitz's portfolio.

APPENDIX

A. Proof of Lemma 5

First observe that if $i \leq j$, then it must be that $w_i^\star \geq w_j^\star$. Because if not, we can simply swap them without increasing the objective value. If there exists $w_i^\star > 0$, then $w_j^\star = t$ satisfies for $\forall j < i$. Because if there exist $w_j^\star < t$, we can decrease $w_i^\star$ and add the same increment to $w_j^\star$ until $w_i^\star = 0$ or $w_j^\star = t$. The constraints are not violated, while the objective function is not decreasing. Similarly, if there exists $w_i^\star < 0$, then $w_j^\star = -t$ satisfies for $\forall j > i$. Therefore, the optimal solution must admit the certain form as $[t, \ldots, t, w_i^\star, 0, \ldots, 0, w_j^\star, -t, \ldots, -t]^T$, where $0 \leq w_i^\star \leq t$ and $-t \leq w_j^\star \leq 0$. Considering the constraint $\mathbf{1}^T\mathbf{w} = 0$, we have $w_i^\star = -w_{N+1-i}^\star$ satisfied. Besides, if $i \leq \left\lfloor \frac{N+1}{2} \right\rfloor$, meaning that $i \leq N+1-i$, then it must be $w_i^\star \geq w_{N+1-i}^\star$. Using $w_i^\star = -w_{N+1-i}^\star$ and $w_i^\star \geq w_{N+1-i}^\star$, we can easily get $w_i^\star \geq 0$.

B. Proof of Lemma 9

The Lagrangian of problem (21) is given by:

$$L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \eta) = \hat{\boldsymbol{\mu}}^T\mathbf{w} - \epsilon t + \boldsymbol{\alpha}^T\mathbf{w} + \boldsymbol{\beta}^T\left(t\mathbf{1} - \hat{\boldsymbol{\sigma}} \odot \mathbf{w}\right) + \eta\left(1 - \mathbf{1}^T\mathbf{w}\right) \quad (39)$$

Then, we write the KKT conditions as:

$$\begin{aligned}
&\mathbf{1}^T\mathbf{w} = 1,\ \mathbf{0} \leq \hat{\boldsymbol{\sigma}} \odot \mathbf{w} \leq t\mathbf{1}, \\
&\boldsymbol{\alpha} \geq 0,\ \boldsymbol{\beta} \geq 0, \\
&\boldsymbol{\alpha}^T\mathbf{w} = \mathbf{0},\ \boldsymbol{\beta}^T\left(t\mathbf{1} - \hat{\boldsymbol{\sigma}} \odot \mathbf{w}\right) = \mathbf{0}, \\
&\frac{\partial L}{\partial \mathbf{w}} = \hat{\boldsymbol{\mu}} + \boldsymbol{\alpha} - \boldsymbol{\beta} \odot \hat{\boldsymbol{\sigma}} - \eta\mathbf{1} = 0, \\
&\frac{\partial L}{\partial t} = \mathbf{1}^T\boldsymbol{\beta} - \epsilon = 0.
\end{aligned} \quad (40)$$

According to complementary slackness, if $w_i > 0$, then $\alpha_i = 0$ and $\beta_i = (\hat{\mu}_i - \eta)/\hat{\sigma}_i \geq 0$, and further $w_i = t/\hat{\sigma}_i$ when $\hat{\mu}_i > \eta$. In contrast, if $w_i = 0$, then $\beta_i = 0$ (since it must be that $t > 0$) and $\alpha_i = \eta - \hat{\mu}_i \geq 0$. Equivalently, we can write the following expressions for α_i, β_i, and w_i w.r.t. η as:

$$\beta_i = \frac{(\hat{\mu}_i - \eta)^+}{\hat{\sigma}_i}, \quad w_i = \begin{cases} t/\hat{\sigma}_i & \hat{\mu}_i > \eta \\ 0 \leq w_i \leq t/\hat{\sigma}_i & \hat{\mu}_i = \eta \\ 0 & \hat{\mu}_i < \eta \end{cases}. \quad (41)$$

Then, we can easily apply $\mathbf{1}^T\boldsymbol{\beta} = \sum_{i=1}^N (\hat{\mu}_i - \eta)^+/\hat{\sigma}_i = \epsilon$ to find the $\eta^\star$. According to the Assumption 3, we have $\hat{\mu}_i \neq \eta^\star$, which means $w_i^\star = \{0, t/\hat{\sigma}_i\}$ is always satisfied. Recall that $\hat{\mu}_1 \geq \hat{\mu}_2 \geq \cdots \geq \hat{\mu}_N$, then the $\mathbf{w}^\star = [t/\hat{\sigma}_1, \ldots, t/\hat{\sigma}_i, 0, \ldots, 0]^T$ is the optimal solution to problem (9). Denote by m the number of non-zero elements in $\mathbf{w}^\star$, it is easy to find that m is equal to the maximum $k \in \{1, \ldots, N\}$ satisfying $\sum_{i=1}^k (\hat{\mu}_i - \hat{\mu}_k)/\hat{\sigma}_i < \epsilon$. Finally considering $\mathbf{1}^T\mathbf{w} = 1$, we then have $t = 1/\sum_{i=1}^m \hat{\sigma}_i^{-1}$.

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